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A POLYNOMIAL RATE OF GROWTH FOR THE MULTIPLICITIES IN COCHARACTERS OF MATRICES

BY

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ABSTRACT

The sum of the multiplicities m_λ, as well as each m_λ, in the cocharacter $\chi_n(F_k)$ of the $k \times k$ matrices, have an upper bound of a polynomial rate of growth. Some have a lower bound which is also of a polynomial rate of growth.

Introduction

Let F be a field of characteristic zero, F_k the $k \times k$ matrices over F and let $\chi_n(F_k) = \sum_{\lambda \in \Lambda_{k^2}(n)} m_\lambda \chi_\lambda$ be its n-th cocharacter [4]. The inequality $m_\lambda \geqq (w_1 + 1)(w_2 - 1)(w_3 + 1)$ [4, theorem 3.22], proved for F_2, yields a lower bound for m_λ which, as a function of n, has a polynomial rate of growth. Thus most m_λ's, and therefore also $\sum_\lambda m_\lambda$, have a lower bound of a polynomial rate of growth.

The purpose of this paper is to produce, for any F_k, upper and lower bounds of similar nature for $\sum_\lambda m_\lambda$. Thus the main result here is Theorem 4.5, which does exactly this.

Section 2 studies and compares linearization with substitutions, as FS_n right module homomorphism. Sections 1, 3 generalize and strengthen [4, §1, 3]. In particular, Theorem 3.1 characterizes the m_λ's in a cocharacter. These results, together with [6, theorem 8] and with the asymptotic results of [5] and [7], allow us to prove our main result.

§1. Adding tableaux

The "gluing together" of two tableaux [4, §1] is now generalized.

Let $\lambda = (a_1, \cdots, a_r) \in \mathrm{Par}(m)$ and write

$$\lambda = (w_1 + \cdots + w_r, w_2 + \cdots + w_r, \cdots, w_r):$$

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the Young diagram D_λ is made of $j \times w_j$ rectangles. To emphasize the "block" structure of λ we introduce the notation

$$\lambda = (w_1 + \cdots + w_r, \cdots, w_r) \overset{\text{def}}{=} \langle w_r, \cdots, w_1 \rangle \qquad \left(m = \sum_{j=1}^{r} j w_j \right).$$

EXAMPLE. $\mathrm{Par}(12) \ni \lambda = (5,3,3,1) = \langle 1,2,0,2 \rangle$.

Note: Some of the w_j's, in particular w_r, can be zero. Also, $\langle w_r, \cdots, w_1 \rangle = \langle 0, \cdots, 0, w_r, \cdots, w_1 \rangle$. Thus, given $\lambda \in \mathrm{Par}(m)$, $\mu \in \mathrm{Par}(n)$, we can write $\lambda = \langle w_r, \cdots, w_1 \rangle$ and $\mu = \langle w'_r, \cdots, w'_1 \rangle$. One can then add λ and μ.

DEFINITION 1.1. Let $\lambda = \langle w_r, \cdots, w_1 \rangle \in \mathrm{Par}(m)$, $\mu = \langle w'_r, \cdots, w'_1 \rangle \in \mathrm{Par}(n)$ and define $\lambda + \mu \in \mathrm{Par}(m+n)$ by $\lambda + \mu = \langle w_r + w'_r, \cdots, w_1 + w'_1 \rangle$. Let $m \leqq l$, $\lambda \in \mathrm{Par}(m)$, $\nu \in \mathrm{Par}(l)$. We say that $\lambda \leqq \nu$ if $\nu = \lambda + \mu$ for some $\mu \in \mathrm{Par}(l-m)$. Equivalently,

$$\langle w_r, \cdots, w_1 \rangle \leqq \langle v_r, \cdots, v_1 \rangle \qquad \text{if } w_j \leqq v_j, \quad j = 1, \cdots, r.$$

Two arbitrary tableaux T_λ, T_μ can be added to yield a tableau on $\lambda + \mu$. The relation between the three corresponding idempotents is then studied.

DEFINITION 1.2. Let $\lambda = \langle w_r, \cdots, w_1 \rangle \in \mathrm{Par}(m)$ and T_λ a corresponding tableau. The block of columns of height j in T_λ forms a $j \times w_j$ rectangle which we denote by B_j ($B_j = \emptyset$ if $w_j = 0$). Now write $T_\lambda \overset{\text{def}}{=} \langle B_r, B_{r-1}, \cdots, B_1 \rangle$. Let $T_\mu = \langle B'_r, \cdots, B'_1 \rangle$, $\mu \in \mathrm{Par}(n)$ be a second tableau. The entries of $T_\mu + m = \langle B'_r + m, \cdots, B'_1 + m \rangle$ are $\{m+1, \cdots, m+n\}$, hence disjoint from those of T_λ, i.e. from $\{1, \cdots, m\}$. Thus

$$T_\lambda + (T_\mu + m) \overset{\text{def}}{=} \langle B_r \,|\, (B'_r + m), \cdots, B_1 \,|\, (B'_1 + m) \rangle$$

is a tableau on $\lambda + \mu$.

EXAMPLE. $T_\lambda = \begin{array}{|c|c|c|} \hline 4 & 2 & 3 \\ \hline 5 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$, $B_1 = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}$, $B_2 = \emptyset$, $B_3 = \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline 1 \\ \hline \end{array}$.

Let $T_\mu = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 \\ \cline{1-1} \end{array}$, then $T_\mu + 5 = \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 6 \\ \cline{1-1} \end{array}$ and $T_\lambda + (T_\mu + 5) = \begin{array}{|c|c|c|c|c|} \hline 4 & 7 & 2 & 3 & 8 \\ \hline 5 & 6 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$.

Let $\kappa(T_\nu) = \{\kappa_1, \cdots, \kappa_a\}$ denote the set of columns of the tableau T_ν, $\nu \in \mathrm{Par}(k)$. Let $S(\kappa_i) \subseteq S_k$ denote the permutations of its entries, so $C_{T_\nu} = \times_{i=1}^{a} S(\kappa_i)$. Clearly, $\kappa(T_\lambda + (T_\mu + m)) = A_1 \cup A_2$ (disjoint union) with $A_1 =$

$\kappa(T_\lambda)$ (entries: $1,\cdots,m$) and $A_2=\kappa(T_\mu+m)$ (entries: $m+1,\cdots,n$). Thus $C_{T_\lambda+(T_\mu+m)}=C_{T_\lambda}\times C_{(T_\mu+m)}$. As in [4, 1.4], this implies that

$$g_{T_\lambda+(T_\mu+m)}(x_1,\cdots,x_{m+n})=g_{T_\lambda}(x_1,\cdots,x_m)\cdot g_{T_\mu}(x_{m+1},\cdots,x_{m+n}).$$

With the proofs of [4, 1.5, 1.6] unchanged, we now have the following, more general

THEOREM 1.3. *Let* $\lambda\in\mathrm{Par}(m)$, $\mu\in\mathrm{Par}(n)$, *then*

$$e_{T_\lambda+(T_\mu+m)}(y)=d\cdot e_{T_\lambda}(y)\cdot e_{T_\mu}(y)$$

for some (integer) $d\geqq 1$.

COROLLARY 1.4. *Let* $\lambda\in\mathrm{Par}(m)$, $T_1,\cdots,T_s$ *s tableaux on* D_λ *with corresponding (semi-) idempotents* $e_1,\cdots,e_s$. *Let* $\mu\in\mathrm{Par}(n)$, T_μ *(one) tableau with* $e_{T_\mu}=e_\mu$. *Construct the s tableaux* $T_j+(T_\mu+m)$ *and let* $\hat{e}_j$ *be their corresponding (semi-) idempotents,* $j=1,\cdots,s$. *Then for some* $d>0$ *(integer),* $\hat{e}_j(y)=d\cdot e_j(y)\cdot e_\mu(y)$, $j=1,\cdots,s$.

NOTE. In §3 of [4], instead of writing $S^{T_\lambda}_{(x,y)}(g_{T_\lambda}(x))=g_{T_\lambda}(y)$, the notation is changed to $S^{T_\lambda}_{(x,y)}(g_{T_\lambda}(x))=p_{T_\lambda}(y)$. Thus, for some $d>0$, $\hat{p}_j(y)=d\cdot p_j(y)\cdot p_\mu(y)$, $j=1,\cdots,s$.

§2. Identifications $S^{T_\lambda}_{(x,y)}$ versus linearization

The substitution $S^{T_\lambda}_{(x,y)}$, [4, §3], identifies the x's in the i-th row of T_λ to y_i. Linearization turns out to be an inverse operation to $S^{T_\lambda}_{(x,y)}$, as we now show.

The symmetric group S_n acts from the right on (any) monomial of degree n by permuting places: $\eta\in S_n$

$$(y_{i_1}\cdots y_{i_n})\eta=y_{i_{\eta(1)}}\cdots y_{i_{\eta(n)}}.$$

This action extends linearly to homogeneous polynomials of degree n. Denote by L the linearization operator. It is determined up to a choice of "names" for the new variables. We choose them so as to end in FS_n. L has the following property.

L.2.1. If $q(y_1,\cdots,y_n)$ is an identity of some algebra A, then $L(q)$ is an identity of A.

L is defined on monomials, then extended by linearity to polynomials. Let $M(y_1,\cdots,y_h)$ be any monomial of degree n and let $a_i=\deg_{y_i}(M)$, then there exists $\eta\in S_n$ such that

$$M(y_1, \cdots, y_h) = (y_1^{a_1} \cdots y_h^{a_h})\eta.$$

Obviously the linearization satisfies

L.2.2. $L(M) = L(y_1^{a_1} \cdots y_h^{a_h} \cdot \eta) = (L(y_1^{a_1} \cdots y_h^{a_h})) \cdot \eta.$

In the special case that $M(y) = y^n$, it is well-known that $L(y^n) = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$ $(\equiv \sum_{\sigma \in S_n} \sigma)$. In the general case, $L(y_1^{a_1} \cdots y_h^{a_h}) = L_1(y_1^{a_1}) \cdots L_h(y_h^{a_h})$, where the x-variables used by L_u and L_v are disjoint when $u \neq v$. Choose

$$x_1, \cdots, x_{a_1} \qquad \text{for } L_1(y_1^{a_1}),$$

$$x_{a_1+1}, \cdots, x_{a_1+a_2} \quad \text{for } L_2(y_2^{a_2}), \quad \text{etc.},$$

i.e.

$$L_1(y_1^{a_1}) = \Big(\sum_{\sigma \in S_{a_1}(a, \cdots, a_1)} \sigma \Big) x_1 \cdots x_{a_1},$$

$$L_2(y_2^{a_2}) = \Big(\sum_{\sigma \in S_{a_2}(a_1+1, \cdots, a_1+a_2)} \sigma \Big) x_{a_1+1} \cdots x_{a_1+a_2}, \quad \text{etc.}$$

Thus, if

$$R_{(a)} = R_{(a_1, \cdots, a_h)} = S_{a_1}(1, \cdots, a_1) \times S_{a_2}(a_1+1, \cdots, a_1+a_2) \times \cdots$$

(a Young subgroup)

and $\bar{R}_{(a)} = \sum_{\sigma \in R_{(a)}} \sigma$, then

L.2.3. $L(y_1^{a_1} \cdots y_h^{a_h}) = \bar{R}_{(a)} (\equiv \bar{R}_{(a)} \cdot x_1 \cdots x_n)$ and $L(y_1^{a_1} \cdots y_h^{a_h} \cdot \eta) = \bar{R}_{(a)} \cdot \eta.$

NOTE. Even though η is not unique, L is independent of its choice: $y_1^{a_1} \cdots y_h^{a_h} \cdot \eta = y_1^{a_1} \cdots y_h^{a_h} \cdot \tau \Leftrightarrow \eta\tau^{-1} \in R_{(a)} \Leftrightarrow R_{(a)}\eta = R_{(a)}\tau \Leftrightarrow \bar{R}_{(a)}\eta = \bar{R}_{(a)}\tau.$

Linearity, L.2.2 and L.2.3 imply:

L.2.4. Let $q(y_1, \cdots, y_h)$ be homogeneous in each y_i and let $g \in FS_n$, then $L(q(y) \cdot g) = (L(q(y)))g$. In particular, let $\deg_{y_i}(q) = a_i$, so there is a $g \in FS_n$ such that

$$q(y) = y_1^{a_1} \cdots y_h^{a_h} \cdot g,$$

so $L(q(y)) = (L(y_1^{a_1} \cdots y_h^{a_h}))g = \bar{R}_{(a)} \cdot g.$

Now turn to substitutions: Let $S_{(x,y)}(x_j) = y_{i_j}$, $1 \leq j \leq n$, $1 \leq i_j \leq h$, so $S_{(x,y)}(x_1 \cdots x_n) = y_{i_1} \cdots y_{i_n} = y_1^{a_1} \cdots y_h^{a_h} \cdot \eta$. Thus $S_{(x,y)}$ determines $a_1, \cdots, a_h$ (hence $R_{(a)} \subseteq S_n$) and the coset $R_{(a)}\eta$. (Clearly, $S_{(x,y)} \leftrightarrow ((a), R_{(a)}\eta)$ classifies all substitutions.) It is easy to check that $S_{(x,y)}(x_{\sigma(1)} \cdots x_{\sigma(n)}) = y_{i_1} \cdots y_{i_n} \cdot \sigma$, so, by linearity we have

S.2.5. Let $g \in FS_n \equiv V_n$, then

$$S_{(x,y)}(g) = (S_{(x,y)}(x_1 \cdots x_n)) \cdot g = y_1^{a_1} \cdots y_h^{a_h} \eta \cdot g.$$

This clearly implies $S_{(x,y)}(g_1 g_2) = (S_{(x,y)}(g_1))g_2$, $g_1, g_2 \in FS_n$.

The substitutions $S_{(x,y)} = S^{T_\lambda}_{(x,y)}$ are defined in [4, §3]. Since

$$a_i = \deg_{y_i}(S^{T_\lambda}_{(x,y)}(x_1 \cdots x_n)) = \text{length of the } i\text{-th row of } T_\lambda,$$

hence $a_1 \geqq \cdots \geqq a_h$. The converse is

LEMMA 2.6. *Let* $S_{(x,y)}(x_1 \cdots x_n) = y_1^{a_1} \cdots y_h^{a_h} \cdot \eta$ *and assume* $a_1 \geqq \cdots \geqq a_h$. *Let* $\lambda = (a_1, \cdots, a_h)$, *so* $\lambda \in \mathrm{Par}(n)$, *and define the tableau* $T_{\lambda,0} = T_0$ *on* λ *as follows*:

$$T_0 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & \cdots & & \cdots & a_1 \\ \hline a_1+1 & a_1+2 & \cdots & a_1+a_2 & & \\ \hline \vdots & \vdots & & & \iddots & \\ \hline \cdot & \cdot & \cdots & & & \\ \hline \end{array}$$

then $S_{(x,y)} = S^{\eta^{-1}T_0}_{(x,y)}$.

PROOF. By definition,

$$S^{\eta^{-1}T_0}_{(x,y)} : x_{\eta^{-1}(1)}, \cdots, x_{\eta^{-1}(a_1)} \rightarrow y_1$$

$$x_{\eta^{-1}(a_1+1)}, \cdots, x_{\eta^{-1}(a_1+a_2)} \rightarrow y_2$$

etc.

Since $S_{(x,y)}(x_1 \cdots x_n) = y_1^{a_1} \cdots y_h^{a_h} \cdot \eta$, hence $y_1^{a_1} \cdots y_h^{a_h} = S_{(x,y)}(x_1 \cdots x_n)\eta^{-1} = S_{(x,y)}(x_{\eta^{-1}(1)} \cdots x_{\eta^{-1}(n)}) = (S_{(x,y)}(x_{\eta^{-1}(1)})) \cdot \cdots \cdot (S_{(x,y)}(x_{\eta^{-1}(n)}))$. Compare places in the two monomials to verify that $S_{(x,y)} = S^{\eta^{-1}T_0}_{(x,y)}$. Q.E.D.

NOTE. $\lambda \in \mathrm{Par}(n)$ determines $T_0 = T_{\lambda,0}$. Given a second tableau T_λ (on λ), there exists (unique) $\eta \in S_n$ such that $\eta T_\lambda = T_0$.

LEMMA 2.7. $L(S^{T_\lambda}_{(x,y)}(x_1 \cdots x_n)) = \eta \bar{R}_{T_\lambda}$, *where* $\eta T_\lambda = T_0$.

PROOF. $T_\lambda = \eta^{-1} T_0$, so by 2.6, $S^{T_\lambda}_{(x,y)}(x_1 \cdots x_n) = y_1^{a_1} \cdots y_h^{a_h} \cdot \eta$, hence $L(S^{T_\lambda}_{(x,y)}(x_1 \cdots x_n)) = L(y_1^{a_1} \cdots y_h^{a_h})\eta = \bar{R}_{(a)}\eta = \bar{R}_{T_0} \cdot \eta$. Now $T_0 = \eta T_\lambda$, so $R_{T_0} = \eta R_{T_\lambda} \eta^{-1}$, and $R_{T_0}\eta = \eta R_{T_\lambda}$, hence $\bar{R}_{T_0}\eta = \eta \bar{R}_{T_\lambda}$. Q.E.D.

The tableau T_λ defines the subgroups $R_{T_\lambda}, C_{T_\lambda} \subseteq S_n$. Denote $\Sigma_{\sigma \in R_{T_\lambda}} \sigma = \bar{R}_T$, $\Sigma_{\sigma \in C_{T_\lambda}} (-1)^\sigma \sigma = \bar{C}_{T_\lambda}$, so $e_{T_\lambda} = \bar{R}_{T_\lambda} \cdot \bar{C}_{T_\lambda}$. By [4, §1,3], $\bar{C}_{T_\lambda} = g_{T_\lambda}$ and $S^{T_\lambda}_{(x,y)}(g_{T_\lambda}) = p(y_1, \cdots, y_h)$.

THEOREM 2.8. *Let* $\eta T_\lambda = T_0$ *(as above) and write* $S_{(x,y)}^{T_\lambda}(g_{T_\lambda}(x_1,\cdots,x_n)) = p(y_1,\cdots,y_h)$, *then* $L(p(y_1,\cdots,y_h)) = \eta e_{T_\lambda}$.

PROOF.

$$L(p(y)) = L(S_{(x,y)}^{T_\lambda}(g_{T_\lambda})) \underset{[2.4,2.5]}{=\!=\!=} (L(S_{(x,y)}^{T_\lambda}(x_1\cdots x_n)))g_{T_\lambda}$$

$$\underset{[2.7]}{=\!=} \eta \bar{R}_{T_\lambda}\cdot g_{T_\lambda} = \eta e_{T_\lambda}.$$

REMARK 2.9. $F\langle y_1,\cdots,y_h\rangle \supset W_n$ = the homogeneous polynomials of degree n, as FS_n right module. Given $a_1+\cdots+a_h = n$, let $W_{(a)} = W_{(a_1,\cdots,a_h)}$ be the FS_n submodule of the polynomials $q(y)$ homogeneous in each y_i, and $\deg_{y_i} q = a_i$, then $W_n = \bigoplus_{(a)} W_{(a)}$. By L.2.4 and S.2.5, $L: W_{(a)} \to V_n \equiv FS_n$ and $S_{(x,y)}: V_n \to W_{(a)}$ $(S_{(x,y)} \leftrightarrow ((a), R_{(a)}\eta))$ are module homomorphisms. Some compositions $L\circ S_{(x,y)}$ are calculated by 2.8. One can easily calculate $S_{(x,y)}\circ L$.

§3. A characterization of m_λ

The following part strengthens [4, theorem 3.5] to a characterization of the m_λ's in a cocharacter. Let A be a P.I. algebra, $Q = I(A) \subset F\langle x\rangle$ its identities and $\chi_n(A) = \sum_{\lambda\in \mathrm{Par}(n)} m_\lambda \chi_\lambda$ its n-th cocharacter. To a tableau T_λ (on λ) corresponds $e_{T_\lambda} = e_{T_\lambda}(x_1,\cdots,x_n)$, and by [4, 1.3], $S_{(x,y)}^{T_\lambda}(e_{T_\lambda}(x)) = |R_{T_\lambda}|\cdot p_{T_\lambda}(y_1,\cdots,y_h)$ $(h = h(\lambda))$.

THEOREM 3.1. *With the above notations,* m_λ *equals the maximal number of (standard) tableaux* $T_1,\cdots,T_s$ *(on* λ*) with* $T_i \leftrightarrow e_i = e_i(x)$, $S_{(x,y)}^{T_i}(e_i(x)) = |R_{T_i}|\cdot p_i(y)$, *such that* $p_1(y),\cdots,p_s(y)$ *are linearly independent modulo* Q *(in* $F\langle x\rangle$*).*

PROOF. (a) (The proof of this part is almost identical to that of [4, theorem 3.5].) Assume $T_1,\cdots,T_s$ are tableaux such that their corresponding $p_1(y),\cdots,p_s(y)$ are linearly independent modulo Q. If we show that

$$\sum_{i=1}^{s} FS_n e_i = \bigoplus_{i=1}^{s} FS_n e_i \quad\text{and}\quad \left(\bigoplus_{i=1}^{s} FS_n e_i\right)\cap Q = 0,$$

then $m_\lambda \geqq s$. Both follow once we prove: If $\sum_{i=1}^s b_i e_i \in Q$, $b_1\in FS_n$, then $b_i e_i = 0$, $i = 1,\cdots,s$. So, let $\sum_{i=1}^s b_i e_i \in Q$ and assume $b_1 e_1 \neq 0$. There exists $c\in FS_n$ such that $cb_1 e_1 = e_1$. Write $c_i = cb_i$, $i = 2,\cdots,s$, then $e_1 + c_2 e_2 + \cdots + c_s e_s \in Q$. Apply $S_{(x,y)}^{T_1}$ and [4, 3 4] to obtain $|R_{T_1}|\cdot p_1(y) + \alpha_2 p_2(y) + \cdots + \alpha_s p_s(y) \in Q$, contradicting the assumed linear independence modulo Q. [Note: No assumption is made of T_i being standard.]

(b) Write $m_\lambda = s$. Let $I_\lambda \subseteq FS_n$ be the two-sided ideal corresponding to λ, and write $Q_\lambda = I_\lambda \cap Q$. By the theory of S_n-representations and by [3, ch. IV, theorem 1] there are s (standard) tableaux $T_1, \cdots, T_s$ (on λ) with corresponding $e_1, \cdots, e_s$ such that

$$(*) \qquad I_\lambda = Q_\lambda \oplus \left(\bigoplus_{i=1}^{s} FS_n e_i \right).$$

Write $S^{T_\lambda}_{(x,y)}(e_i) = |R_{T_i}| \cdot p_i(y)$, $1 \leqq i \leqq s$, and assume $\Sigma_{i=1}^{s} \alpha_i p_i(y) \in Q$. Note that all $p_i(y)$ belong to the same $W_{(a)}$. Apply L! By $L.2.1$ and Theorem 2.8, $\Sigma_{i=1}^{s} \alpha_i \eta^{(i)} e_i \in Q$ for some $\eta^{(1)}, \cdots, \eta^{(s)} \in S_n$, so by $(*)$, $\alpha_1 = \cdots = \alpha_s = 0$.

Q.E.D.

The last theorem has the following application for F_2:

Lemma 3.2. *Let $\lambda \in \mathrm{Par}(n)$, $\lambda' \in \mathrm{Par}(n')$, $n \leqq n'$, $\lambda = \langle w_4, \cdots, w_1 \rangle \leqq \lambda' = \langle w'_4, \cdots, w'_1 \rangle$ (Definition 1.1) with $w_4 + 2 \leqq w'_4$, then $m_\lambda \leqq m_{\lambda'}$.*

Proof. Write $m_\lambda = s$, so there are s tableaux $T_1, \cdots, T_s$ (on λ) with their corresponding $p_1(y), \cdots, p_s(y)$ linearly independent modulo Q. Let $\mu = \langle w'_4 - w_4, \cdots, w'_1 - w_1 \rangle$. By the results of [4, §3] there exists a tableau T_μ (on μ) whose corresponding $p_\mu(y)$ is an F_2-non-identity.

Let $T'_j = T_j + (T_\mu + n)$, $j = 1, \cdots, s$. T'_j are tableaux on $\lambda + \mu = \lambda'$, with corresponding $p'_j(y)$. By Corollary 1.4 there exists $d > 0$ such that

$$p'_j(y) = dp_j(y) \cdot p_\mu(y), \qquad j = 1, \cdots, s.$$

By the primeness property of $Q = I(F_2)$, [1], $p'_1(y), \cdots, p'_s(y)$ are linearly independent modulo Q. Therefore by Theorem 3.1 ($m_\lambda =$) $s \leqq m_{\lambda'}$. Q.E.D.

§4. A polynomial bound for the m_λ's

Given (large) n, we give one upper bound to all the m_λ's of $\chi_n(F_2)$, a bound which is a polynomial in n. By a tensor product technique, the result is extended to all rings of matrices.

Notation. Given n, let $t = t(n) = 16n^2 + 10n$ and define

$$\mathrm{Par}(t) \ni \mu(n) = (4n^2 + 4n, 4n^2 + 3n, 4n^2 + 2n, 4n^2 + n).$$

Note. $\lambda < \mu(n)$ for any $\lambda \in \mathrm{Par}(n)$, hence for $\chi_n(F_2)$, $m_\lambda \leqq m_{\mu(n)}$. The asymptotic results of [5] are applied to give a lower bound for $d_{\mu(n)}$, then an upper bound for $m_{\mu(n)}$.

Lemma 4.1. *There exist a constant c and a large enough N such that for all $n \geqq N$, $d_{\mu(n)} > c \cdot (1/t)^{9/2} \cdot 4^t$ ($t, \mu(n)$ as above).*

PROOF. Let $a = 1$ and $\delta = 1/8$ in [5, 1.1], then choose N large enough such that F.1.1 (there) works if $t(n) \geqq N$. $N \leqq n$ should also be large enough for the following asymptotic arguments:

Write $\mu(n) = (t/4 + c_1\sqrt{t}, \cdots, t/4 + c_4\sqrt{t})$, so

$$c_j = \frac{5-2j}{2} \cdot \frac{n}{\sqrt{16n^2+10n}} \simeq \frac{5-2j}{8} \quad \text{and} \quad c_j - c_{j+1} \simeq \frac{1}{4}.$$

Thus $\mu(n) \in \Lambda_4(t, 1, \frac{1}{8})$ and by F.1.1,

$$d_{\mu(n)} \simeq \gamma_4 \cdot D(c_1, \cdots, c_4) \cdot e^{-2(c_1^2+\cdots+c_4^2)} \cdot \left(\frac{1}{t}\right)^{9/2} \cdot 4^t.$$

Now refer to [5, note 2.1]: $P_4(1, 1/8)$ is compact, hence $D(c_1, \cdots, D_4)e^{-2(c_1^2+\cdots+c_4^2)}$ has a minimum $M \gneqq 0$ on it. Thus for large enough n, $d_{\mu(n)} > \frac{1}{2}\gamma_4 \cdot M \cdot (1/t)^{9/2} \cdot 4^t$. Choose $c = \frac{1}{2}\gamma_4 M$ to complete the proof. Q.E.D.

LEMMA 4.2. *Let $N = N(1, 1/8)$ as above. For some $d > 0$, $m_{\mu(n)} < dt^3$ for all $t(n)$, $n \geqq N$.*

PROOF. For a large enough t,

$$m_{\mu(n)} \cdot c \cdot \left(\frac{1}{t}\right)^{9/2} \cdot 4^t \underset{\text{[Lemma 4.1]}}{<} m_{\mu(n)} \cdot d_{\mu(n)} \leqq c_t(F_2) \underset{\text{[4, theorem 5.4]}}{\lesssim} \frac{4}{\pi}\left(\frac{1}{t}\right)^{3/2} \cdot 4^t,$$

hence $m_{\mu(n)} < (4/c\sqrt{\pi}) \cdot t^3$. Choose $d = 4/c\sqrt{\pi}$. Q.E.D.

It is well-known that $|\Lambda_h(n)| \leqq \binom{n+h-1}{n} < n^{h-1}$, [2]. By Lemma 3.2 we deduce

COROLLARY 4.3. *If n is large enough, then for all $\lambda \in \mathrm{Par}(n)$, $m_\lambda < d \cdot t^3 = d \cdot (16n^2 + 10n)^3 \leqq d'n^6$, hence $\sum_{\lambda \in \Lambda_4(n)} m_\lambda < |\Lambda_4(n)| \cdot d \cdot (16n^2 + 10n)^3 < d \cdot n^3 \cdot (16n^2 + 10n)^3$: the sum of multiplicities is polynomially bounded.*

REMARK 4.4. Write $n = 6l + r$, $0 \leqq r \leqq 5$, $l = [n/6] \simeq n/6$, and let $\lambda = (3l, 2l, l + r) = \langle w_4, \cdots, w_1 \rangle$, where $w_4 = 0$, $w_3 = w_2 = l$ and $w_1 = l + r$. By [4, theorem 3.22], $m_\lambda \geqq (w_1 + 1)(w_2 - 1)(w_3 + 1) = (l + r + 1)(l - 1)(l + 1) \simeq l^3 \simeq (\frac{1}{6})^3 n^3$, for large n. Thus some (many) m_λ's of $\chi_n(F_2)$ also have a lower bound of a polynomial rate of growth. For $k \geqq 2$, $\chi_n(F_2) \leqq \chi_n(F_k)$, hence the same is true for any $\chi_n(F_k)$.

We now generalize Corollary 4.3 to all matrix rings F_k.

THEOREM 4.5. *Let $\chi_n(F_k) = \sum_{\lambda \in \Lambda_{k^2}(n)} m_\lambda \chi_\lambda$. There exist an exponent $e = e(k)$ and a constant c such that $\sum_{\lambda \in \Lambda_{k^2}(n)} m_\lambda \leqq c \cdot n^e$. In particular, $m_\lambda \leqq c'n^{e'}$ for all $\lambda \in \Lambda_{k^2}(n)$ where $c' > 0$ is another constant and $e' \leqq e$ a smaller exponent.*

PROOF. It is enough to prove for large n.

(a) Assume $k = 2^l$ and prove by induction on l, the case $l = 1$ being Corollary 4.3. Prove for $l + 1$:

$$F_{2^{l+1}} \cong F_{2^l} \otimes F_2 \Rightarrow \chi_n(F_{2^{l+1}}) \leqq \chi_n(F_{2^l}) \otimes \chi_n(F_2). \quad [6, \text{theorem } 8]$$

Induction and Corollary 4.3 imply

$$\chi_n(F_{2^l}) \otimes \chi_n(F_2) \leqq \sum_{\nu \in \Lambda_{4^l}(n)} c_1 n^{e_1} \chi_\nu \otimes \sum_{\mu \in \Lambda_4(n)} d' n^6 \chi_\mu$$

$$= c_1 d' n^{e_1+6} \cdot \sum_{\nu \in \Lambda_{4^l}(n)} \sum_{\mu \in \Lambda_4(n)} \chi_\nu \otimes \chi_\mu.$$

By [7, lemma 1] there are e_2, c_2 such that for all $\nu \in \Lambda_{4^l}(n)$, $\mu \in \Lambda_4(n)$, $\chi_\nu \otimes \chi_\mu \leqq \sum_{\lambda \in \Lambda_{4^{l+1}}(n)} c_2 \cdot n^{e_2} \cdot \chi_\lambda$, hence

$$\chi_n(F_{2^{l+1}}) \leqq c_1 c_2 d' n^{e_1+6} \cdot n^{e_2} \cdot |\Lambda_{4^l}(n)| \cdot |\Lambda_4(n)| \cdot \sum_{\lambda \in \Lambda_{4^{l+1}}(n)} \chi_\lambda.$$

Thus

$$\sum_{\lambda \in \Lambda_{k^2}(n)} m_\lambda = \sum_{\lambda \in \Lambda_{4^{l+1}}(n)} m_\lambda \leqq c_1 c_2 d' \cdot n^{e_1+6} \cdot n^{e_2} \cdot n^{4^l-1} \cdot n^3 \cdot n^{4^{l+1}-1}$$

$$= c \cdot n^e,$$

where $c = c_1 c_2 d'$ and $e = e_1 + 6 + e_2 + 4^l - 1 + 4^{l+1} - 1 + 3$.

(b) The general case follows easily: For k arbitrary, choose l such that $k \leqq 2^l$. Since $F_k \leqq F_{2^l}$, F_k satisfies more identities than F_{2^l}, hence $\chi_n(F_k) \leqq \chi_n(F_{2^l})$.

Q.E.D.

REMARK 4.6. Since $\chi_n(F_2) \leqq \chi_n(F_k)$, Remark 4.4 implies that some m_λ's in $\chi_n(F_k)$ have a lower bound which also has a polynomial rate of growth. Thus $\Sigma_\lambda m_\lambda = L_n(F_k)$, the colength of F_k, has lower and upper bounds of polynomial rate of growth.

We think that the property of a polynomial rate of growth of $L_n(A)$ is shared by many other algebras.

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